

# Statistical Mechanics

Exam Version

8/4-2026

For this exam you can bring one A4 cheatsheet (writing on all sides). You can use a calculator. No communication with other people is allowed during the exam.

Location: Exam Hall 4, Time: 15:00-17:00

Norm:

The answers must be given in English and readable handwriting. The table below shows the number of points to be given for each of the questions. For the exam score the total score,  $M$ , is converted to the mark using the max score ( $M_{\max}=75$ ) according to the formula  $\frac{9M}{M_{\max}} + 1$ . This exam counts for 100% of the grade, but the midterm assignment must be completed to pass the course. If 1/2 bonus point was earned during the problem solving session this is added and if the resulting mark is above 10 the final mark will be 10.

| Subquestion | Q1 | Q2 | Q3 |
|-------------|----|----|----|
| A           | 10 | 5  | 10 |
| B           | 5  | 10 | 5  |
| C           | 5  | 5  | 5  |
| D           | 5  | 5  | 5  |

## 1 The energy distribution in anharmonic systems

The Energy in an anharmonic classical system is described by the Hamiltonian:

$$H = \sum_n^N \frac{p_n^2}{2m} + \frac{kq_n^{2K}}{2}.$$

Here,  $N$  is the number of degrees of freedom.  $m$  is the mass and  $k$  a force constant.  $K$  is a positive integer.  $p_n$  and  $q_n$  are the momentum and position of the  $n$ 'th degree of freedom.

- A. What is the average energy in each degree of freedom? (Explain each step of the derivation.)
- B. Derive the equation for the heat capacity for this system?
- C. Explain what happens when  $K$  goes from 1 (harmonic system) to infinity.
- D. Explain why we can neglect the contribution of quark motion to the heat capacity of a dilute gas of protons at room temperature?

## 2 Magnetism

- A. Consider an electron gas in a two-dimensional ( $L \times L \times d$ ) slab, where  $d$  is much smaller than the thermal wavelength of the electrons. A magnetic field  $H$  is applied perpendicular to the slab. Use that the energy levels are given by  $E_j = 2\mu_0 H(j + 1/2)$  ( $j = 0, 1, 2, \dots$ ), where  $\mu_0 = \frac{e\hbar}{2mc}$  and the degeneracy of each level is  $g = \frac{NH}{H_0}$ , where  $H_0 = \frac{Nhc}{eL^2}$ . How large does the magnetic field have to be to make all electrons fit in the lowest energy level?
- B. How does the internal energy depend on the applied magnetic field when the temperature is taken to be zero?
- C. What is the magnetisation of the system induced by the magnetic field?
- D. Give an example of something that can be measured using this so-called de Haas-van Alphen effect.

### 3 Brownian Motion and Fluctuations

A. A free particle is exhibiting Brownian motion. The dynamics follow the equation:

$$m \frac{d^2 x}{dt^2} = -m\gamma \frac{dx}{dt} + F_{\text{random}}(t). \quad (1)$$

Here the random force is uncorrelated with the position of the particle ( $\langle F_{\text{random}}(t)x(t) \rangle = 0$ ). Show that  $\langle \frac{dx}{dt} x \rangle = C \exp(-\gamma t) + \frac{k_B T}{m\gamma}$ .

B. Find an expression for  $\langle x^2 \rangle$  assuming that it has the initial position  $x_0 = 0$  (so  $\langle x(0) \rangle = 0$  and  $\langle x^2(0) \rangle = 0$ ).

C. Give an expression for the diffusion constant for the above system.

D. Free electrons in a metal wire subjected to an electric field can be described with the equation:

$$m \frac{d^2 x}{dt^2} + m\gamma \frac{dx}{dt} = -eE_0 \cos(\omega t), \quad (2)$$

with electron mass  $m$ , and damping constant  $\gamma$ , when  $N$  electrons are being driven by a force (of an applied electric field) of amplitude  $E_0$  and angular frequency  $\omega$ . The resulting current in a DC field is  $j(t) = Ne^2 \frac{1}{m\gamma} E_0 = \mu E_0$ . Show how the electron mobility,  $\mu$ , relates to the diffusion and explain what more general theorem this is a manifestation of.

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This exam has been drafted by T. L. C. Jansen and verified by  
T. A. Schlathölder.

Date: 17/3-2026

Signature:

A handwritten signature in cursive script, appearing to read "Thomas Jansen".

Date: 17/3-2026

Signature:

A handwritten signature in cursive script, appearing to read "T. A. Schlathölder".

## Potentially useful equations

Maxwell relations:

$$P = - \left( \frac{\partial A}{\partial V} \right)_T$$

$$P = - \left( \frac{\partial U}{\partial V} \right)_S$$

$$V = \left( \frac{\partial H}{\partial P} \right)_S$$

$$V = \left( \frac{\partial G}{\partial P} \right)_T$$

$$S = - \left( \frac{\partial A}{\partial T} \right)_V$$

$$S = - \left( \frac{\partial G}{\partial T} \right)_P$$

$$T = \left( \frac{\partial H}{\partial S} \right)_P$$

$$T = \left( \frac{\partial U}{\partial S} \right)_V$$

$$\mu = \left( \frac{\partial A}{\partial N} \right)_{V,T}$$

$$\mu = \left( \frac{\partial G}{\partial N} \right)_{P,T}$$

The thermal wavelength:

$$\lambda = \sqrt{2\pi\hbar^2/mkT}$$

The Ideal gas law:

$$PV = nRT$$

The van der Waas equation:

$$\left[ P + a \left( \frac{n}{V} \right)^2 \right] \left( \frac{V}{n} - b \right) = RT$$

The Sacktur-Tetrode equation:

$$S = k_B N \left( \log \left[ \frac{V}{N} \left( \frac{4\pi m U}{3h^2 N} \right)^{3/2} \right] + \frac{5}{2} \right)$$

The Clausius-Clapeyron equation:

$$\frac{dP(T)}{dT} = \frac{L}{T\Delta V}$$

The functions for the ideal Fermi gas including the low-fugacity expansions:

$$f_{5/2}(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{5/2}}$$

$$f_{3/2}(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{3/2}} \approx z - \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} - \frac{z^4}{4^{3/2}} + \dots$$

$$z = \frac{\lambda^3}{v} + \frac{1}{2^{3/2}} \left( \frac{\lambda^3}{v} \right)^2 + \dots$$

$$\epsilon_F = \frac{\hbar^2}{2m} \left( \frac{6\pi^2}{gv} \right)^{2/3}$$

$$U = \frac{3}{5} N \epsilon_F \left[ 1 + \frac{5}{12} \pi^2 \left( \frac{kT}{\epsilon_F} \right)^2 + \dots \right]$$

The functions for the ideal Bose gas including the low-fugacity expansions

$$g_{5/2}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}}$$

$$g_{3/2}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{3/2}} \approx z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \frac{z^4}{4^{3/2}} + \dots$$

$$g_{5/2}(1) = \zeta(5/2) = 1.342 \dots$$

$$g_{3/2}(1) = \zeta(3/2) = 2.612 \dots$$

Kramers-Kronig relation:

$$\tilde{\chi}'(\omega) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\tilde{\chi}''(\omega')}{\omega' - \omega}$$

Reverse Kramers-Kronig relation:

$$\tilde{\chi}''(\omega) = -\mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\tilde{\chi}'(\omega')}{\omega' - \omega}$$

Fluctuation-Dissipation theorem:

$$\tilde{C}_{xx}(\omega) = 2k_B T \frac{\tilde{\chi}''(\omega)}{\omega}$$

## Mathematical relations

Definition of the polylogarithm:

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \quad (3)$$

Geometric series valid for  $x < 1$ :

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\exp(-ax) = \sum_{n=0}^{\infty} \frac{(-ax)^n}{n!}$$

$$\int_{-\infty}^{\infty} \exp(-a(x+b)^2) dx = \sqrt{\frac{\pi}{a}}$$

With  $\log(x)$  the natural logarithm is considered. For  $-1 < x < 1$ :

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

$$\int \frac{1}{x-a} dx = \log(x-a)$$

The coth function is defined

$$\coth(x) = \frac{\exp(x) + \exp(-x)}{\exp(x) - \exp(-x)}$$

For  $\coth(x)$  we have (for  $0 < |x| < \pi$ )

$$\coth(x) = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \dots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} - \dots$$

The derivative of  $\coth(x)$  is

$$\frac{d \coth(x)}{dx} = \text{csch}^2(x)$$

The derivative of  $\cosh(x)$  is

$$\frac{d \cosh(x)}{dx} = \sinh(x)$$

The derivative of  $\sinh(x)$  is

$$\frac{d \sinh(x)}{dx} = \cosh(x)$$

The derivative of  $\log(x)$  is

$$\frac{d \log(x)}{dx} = 1/x$$

The hyperbolic cosecant is

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

The Taylor expansion of  $\operatorname{csch}^2(x)$  is

$$\operatorname{csch}^2(x) \approx \frac{1}{x^2} \left( 1 - \frac{1}{3}x^2 + \dots \right)$$

The following integrals may be useful:

$$\int_{-\infty}^{\infty} dx \exp(-ax^2) = \sqrt{\pi/a}$$

$$f(y) = \int_0^y dx x^2 \sqrt{1+x^2} = \frac{1}{3}y^3 \left( 1 + \frac{3}{10}y^2 + \dots \right) \text{ for } y \ll 1$$

$$= \frac{1}{4}y^4 \left( 1 + \frac{1}{y^2} + \dots \right) \text{ for } y \gg 1$$

$$\int_0^{\infty} (2J+1) \exp(-aJ(J+1)) dJ = \frac{1}{a}$$

$$\int_0^{\infty} (4J+1) \exp(-2aJ(2J+1)) dJ = \frac{1}{2a}$$

$$\int_0^{\infty} (4J+3) \exp(-a(2J+1)(2J+2)) dJ = \frac{\exp(-2a)}{2a}$$

$$\int_0^{\infty} \frac{x}{\exp(x)-1} dx = \frac{\pi^2}{6}$$

$$\int_0^{\infty} \frac{x^2}{\exp(x)-1} dx = 2\zeta(3) \approx 2.40411$$

$$\int_0^{\infty} \frac{x^3}{\exp(x)-1} dx = \frac{\pi^4}{15}$$

## Potentially useful constants

$$m_e = 9.10938356 \times 10^{-31} \text{ kg}$$

$$m_n = 1.674929 \times 10^{-27} \text{ kg}$$

$$m_p = 1.672623 \times 10^{-27} \text{ kg}$$

$$N_A = 6.022140857 \times 10^{23} \text{ 1/mol}$$

$$\hbar = 1.0545718 \times 10^{-34} \text{ Js}$$

$$k_B = 1.38064852 \times 10^{-23} \text{ J/K}$$

$$k_B = 8.617333262145 \times 10^{-5} \text{ eV/K}$$

$$c = 2.99792458 \times 10^8 \text{ m/s}$$

$$e = 1.60217662 \times 10^{-19} \text{ C}$$

$$R = 8.31446261815324 \text{ J/(mol K)}$$